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Citation for published version:
Lucietti, J 2018, 'On the nonexistence of extreme anti-de Sitter black rings', *Classical and quantum gravity*, vol. 35, no. 21, 21lT01. https://doi.org/10.1088/1361-6382/aae351

Digital Object Identifier (DOI):
10.1088/1361-6382/aae351

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Classical and quantum gravity

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Download date: 04. May. 2023

On the nonexistence of extreme anti-de Sitter black rings

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Abstract

We prove that five-dimensional extreme anti-de Sitter black ring solutions to the vacuum Einstein equations that admit biaxial symmetry do not exist. This is established by demonstrating the nonexistence of five-dimensional, biaxisymmetric, vacuum near-horizon geometries of extreme horizons with a nonzero cosmological constant and ring topology.

A striking result in higher dimensional General Relativity is the failure of the black hole uniqueness theorem [1]. This was revealed by the existence a black ring solution to the five-dimensional vacuum Einstein equations, i.e. an asymptotically flat black hole solution with a horizon of $S^1 \times S^2$ topology [2]. A basic question is whether such black holes exist in the presence of a cosmological constant. Approximate solutions have been constructed corresponding to thin black rings in anti-de Sitter and de Sitter spacetime, for which the radius of the S^2 is much smaller than the cosmological scale [3]. Furthermore, anti-de Sitter black rings have been constructed numerically in regimes not accessible to such approximations [4]. Curiously, these works indicate the nonexistence of thin anti-de Sitter black rings whose S^2 is much larger than the cosmological scale, in agreement with expectations from the AdS/CFT correspondence (i.e. there are no corresponding fluid configurations in the hydrodynamic regime of the CFT [5]).

There are also a number of nonexistence theorems. It has been shown that supersymmetric anti-de Sitter black ring solutions to minimal gauged supergravity do not exist (Einstein-Maxwell-Λ theory coupled to a Chern-Simons term) [6,7]. Even more surprising, it has recently been proven that extreme de Sitter black rings, possibly coupled to matter obeying the dominant energy condition, do not exist [8,9]. These nonexistence results rely on the fact that for any extreme black hole solution the near-horizon geometry also satisfies the Einstein equation and so horizon topologies and geometries can be classified (and ruled out) independently of the black hole classification problem, see [10] for a review.

In this note we will consider five-dimensional spacetimes that obey the vacuum Einstein equations with a cosmological constant Λ and which contain an extreme (degenerate) Killing horizon. As is well known, the Einstein equation for the near-horizon geometry then reduces to the following

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geometric equation for the Riemannian metric g_{ab} induced on a three-dimensional cross-section S of the horizon

$$R_{ab} = \tfrac{1}{2} h_a h_b - D_{(a} h_{b)} + \Lambda g_{ab} \tag{1}$$

where D_a is the metric connection of g_{ab} and h_a is a 1-form on S. For applications to the study of black holes S is assumed to be a compact manifold (no boundary). The classification of solutions to this horizon equation has been extensively studied [10]. For $\Lambda = 0$ a complete classification was derived for solutions that admit $U(1)^2$-symmetry [11]. The assumption of such biaxial symmetry is compatible with both asymptotically flat and Kaluza-Klein (KK) spacetimes, so this classification captures all the known extreme black hole solutions (Myers-Perry black holes, black rings, KK black holes, black strings) and also revealed there are no other extreme horizons in this class.

The $\Lambda \neq 0$ case was also considered and the classification reduced to a 6th order nonlinear ODE of a single function [11]. Other than the known rotating black hole [12], no other solutions to this ODE were found, so the results in this case were inconclusive. Of course, with hindsight, we now know that for $\Lambda > 0$ there can be no black ring solutions from the recent topology theorems [8, 9]. The purpose of this work is to revisit the $\Lambda \neq 0$ case and point out that there is in fact an elementary proof of the nonexistence of black rings that admit $U(1)^2$-symmetry valid for both $\Lambda > 0$ (de Sitter) and $\Lambda < 0$ (anti-de Sitter). In particular, our main result is that five-dimensional extreme vacuum anti-de Sitter black rings that admit biaxial symmetry do not exist.[1]

We first need to recall the relevant results of [11]. Suppose S is a three-dimensional compact (nontoroidal) manifold with $U(1)^2$-symmetry and let $\eta_i, i = 1, 2$, denote the commuting Killing fields which generate the biaxial symmetry. Then, the orbit space $S/U(1)^2$ is a closed interval and the matrix $\gamma_{ij} \equiv g(\eta_i, \eta_j)$ is rank-1 at the endpoints and rank-2 in the interior (see e.g. [13, 14]). The topology of S is then determined by the null space of γ_{ij} at the two endpoints: $S^1 \times S^2$ (ring) if the null spaces at the endpoints are the same; or locally S^3 otherwise (the latter includes the lens spaces). Now, there exists a globally defined $U(1)^2$-invariant function x such that $\mathrm{d}x = -i_{\eta_1} i_{\eta_2} \epsilon_g$ where ϵ_g is the volume form of g_{ab}. Compactness of S, together with the relation $|\mathrm{d}x|^2 = \det \gamma_{ij}$, implies that x attains precisely one minimum and one maximum, say x_0 and x_1 respectively. Hence, we may use x as a coordinate on the interior of the orbit space and in particular identify the orbit space $S/U(1)^2 \cong [x_0, x_1]$.

Next, one can decompose the 1-form $h = \beta + \mathrm{d}\lambda$ globally on S, where β is a co-closed 1-form and λ is a function which are $U(1)^2$-invariant. One then may use coordinates (x, ϕ^i) on S adapted to the symmetry so $\eta_i = \partial_{\phi_i}$ and show that $\beta = \beta_i(x)\mathrm{d}\phi^i$ and

$$g = \frac{\mathrm{d}x^2}{\det \gamma_{ij}} + \gamma_{ij}(x)\mathrm{d}\phi^i \mathrm{d}\phi^j \ . \tag{2}$$

These coordinates are valid for $x_0 < x < x_1$ and break down at the endpoints where $\det \gamma_{ij} = 0$. In order that the metric extends smoothly onto a compact manifold S we must require $(\mathrm{d}|v_I|)^2 \to 1$ at each endpoint $x \to x_I$, $I = 0, 1$, where $v_I = v_I^i \eta_i$ is the vector which vanishes at $x = x_I$ normalised to be 2π-periodic (i.e. v_I^i is in the null space of $\gamma_{ij}(x_I)$). By an $SL(2, \mathbb{Z})$ transformation we may always set $v_I = \eta_1$ for one I, so $\gamma_{i1}(x_I) = 0$ and $\gamma_{22}(x_I) > 0$. Then, in terms of the the proper distance s from x_I, $s = \int_{x_I}^x \sqrt{g_{xx}}\mathrm{d}x$ (which is a global coordinate on $[x_0, x_1]$), smoothness requires $\gamma_{11} = s^2 + O(s^4)$, $\gamma_{12} = O(s^2)$ and $\gamma_{22} = \gamma_{22}(x_I) + O(s^2)$ as $s \to 0$. It follows that $\sqrt{\det \gamma_{ij}}$ as a function of s has simple zeros at the endpoints.

[1] It is worth emphasising that our result does not invoke any global assumptions on the spacetime and hence is valid for both asymptotically AdS and locally AdS spacetimes.

It proves convenient to introduce the globally defined $U(1)^2$-invariant functions $\Gamma \equiv e^{-\lambda}$ and $Q \equiv \Gamma \det \gamma_{ij}$. Note that $\Gamma > 0$ everywhere and $Q(x) > 0$ for $x_0 < x < x_1$ and vanishes at the endpoints $Q(x_0) = Q(x_1) = 0$. Smoothness then implies that $Q(x)$ as a function of x has simple zeros and thus, since $Q(x) > 0$ in the interior, in particular

$$Q'(x_0) > 0, \quad Q'(x_1) < 0 \ . \tag{3}$$

This is not obvious since x is not a well defined coordinate at the endpoints. To see it, note that in terms of the proper distance s we have $\mathrm{d}x/\mathrm{d}s = \sqrt{\det \gamma_{ij}}$ which, as noted above, has simple zeros at the endpoints. It follows that $x - x_I$ has a double zero at the endpoint x_I and also that Q has double zeros at the endpoints. Therefore, $Q(x)$ as a function of x has simple zeros, as claimed.

In [11] it was shown that, for any solution to (1) of the above form, the pair of functions $(\Gamma(x), Q(x))$ obey the ODE system

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q^3}{\Gamma}\frac{\mathrm{d}^3\Gamma}{\mathrm{d}x^3}\right) = 10\Lambda Q^2 \frac{\mathrm{d}^2\Gamma}{\mathrm{d}x^2} \tag{4}$$

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}x^2} + 2C^2 + 6\Lambda\Gamma = 0 \tag{5}$$

where $C > 0$ is a constant (this constant can be set to any value by certain scalings). Furthermore, the full solution may be reconstructed from this data:

$$g = \frac{\Gamma}{Q}\mathrm{d}x^2 + P(\mathrm{d}\phi^1 + \omega \mathrm{d}\phi^2)^2 + \frac{Q}{\Gamma P}(\mathrm{d}\phi^2)^2 \ , \tag{6}$$

where the other metric components are determined by

$$P = \Gamma \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q\Gamma'}{\Gamma}\right) + 2\Gamma(C^2 + \Lambda\Gamma) \tag{7}$$

$$P^2\omega'^2 = \frac{1}{\Gamma}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{QP'}{P}\right) + 2\Lambda + \frac{P}{\Gamma^2} \tag{8}$$

and $f' \equiv \mathrm{d}f/\mathrm{d}x$. It was also shown that

$$\omega' = \frac{k}{P^2\Gamma} \tag{9}$$

where k is a constant. Note that to derive the above a $GL(2, \mathbb{R})$ transformation $\eta_i \to A_{ij}\eta_j$ was performed to align $k^i \equiv \Gamma\gamma^{ij}\beta_j = \delta^i_1$ (constancy of k^i follows from the (xi) component of (1)). Thus, in these coordinates the η_i may not have closed orbits. The boundary conditions for the above ODE system at the endpoints $x = x_I$ are fixed by the requirement that the metric extends to a smooth metric on a compact S, as discussed above. A key consequence of this, which we will use below, is that for any smooth $U(1)^2$-invariant function f on S the function $Qf' = \Gamma\sqrt{\det\gamma_{ij}}\frac{\mathrm{d}f}{\mathrm{d}s}$ is smooth and vanishes at the endpoints (recall s is the proper distance as above).

For $\Lambda = 0$, the system is easy to solve [11]. Then, $Q^3\Gamma'''/\Gamma$ is a constant and evaluating this at the endpoints implies this must vanish everywhere.[2] Thus $\Gamma''' = 0$, so Γ and Q are both quadratic functions of x and the horizon metric is fully determined, leading to a complete classification. We

[2]Again, since x is not a valid coordinate at the endpoints, one must take care here. One can see that $Q^3\Gamma'''$ vanishes at the endpoints by writing it in terms of the proper distance s.

wish to solve the $\Lambda \neq 0$ system. Clearly $\Gamma(x)$ linear solves the system; this gives the known rotating black hole with a cosmological constant [11]. But are there any other solutions? As mentioned above, for $\Lambda > 0$ it has been argued on general grounds that there can't be any ring topology horizons [8]. However, the situation for $\Lambda < 0$ has remained unclear.

We now turn to the $\Lambda \neq 0$ case. Observe that $P = |\eta_1|^2$ and hence $P \geq 0$ and may vanish only at the endpoints. There are a number of cases depending on which Killing field vanishes at the endpoints. First suppose $P > 0$ everywhere on S. Then the Killing field

$$v_I = \eta_2 - \omega(x_I)\eta_1 \tag{10}$$

vanishes at $x = x_I$ for $I = 0, 1$, respectively. If $k \neq 0$ equation (9) implies ω is a monotonic function so that $\omega(x_0) \neq \omega(x_1)$. Thus $v_0 \neq v_1$ and hence the topology in this case must be locally S^3. Therefore, for ring topology we must have $k = 0$ and hence $\omega' = 0$. Then (8) gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{QP'}{P}\right) + 2\Lambda\Gamma + \frac{P}{\Gamma} = 0 \tag{11}$$

and integrating this over the interval we find the boundary term vanishes giving

$$\int_{x_0}^{x_1} \frac{P}{\Gamma}\mathrm{d}x = -2\Lambda \int_{x_0}^{x_1} \Gamma \mathrm{d}x \ . \tag{12}$$

Notice for $\Lambda \geq 0$ we immediately get a contradiction. For $\Lambda < 0$ we need to work a little harder. Integrating (7) divided by Γ gives

$$\int_{x_0}^{x_1} \frac{P}{\Gamma}\mathrm{d}x = \int_{x_0}^{x_1} (2C^2 + 2\Lambda\Gamma)\mathrm{d}x \tag{13}$$

and equating this with (12) gives

$$\int_{x_0}^{x_1} (C^2 + 2\Lambda\Gamma)\mathrm{d}x = 0 \ . \tag{14}$$

On the other hand, integrating (5) gives

$$Q'(x_1) - Q'(x_0) = -2\int_{x_0}^{x_1} (C^2 + 3\Lambda\Gamma)\mathrm{d}x = -2\Lambda \int_{x_0}^{x_1} \Gamma \mathrm{d}x \tag{15}$$

where in the second equality we used (14). From the boundary conditions (3) the LHS is negative. However, for $\Lambda \leq 0$ the RHS is nonnegative, giving a contradiction. We conclude that there are no solutions for *any* Λ with ring topology in the case P is strictly positive (this is consistent with the $\Lambda = 0$ results of [11]).

On the other hand, now suppose P vanishes at one endpoint. Then, ring topology requires it vanishes at both endpoints (it must be the same Killing field vanishing) so $P(x_0) = P(x_1) = 0$, i.e. η_1 vanishes at the endpoints. As discussed above, smoothness then requires that $(\mathrm{d}|\eta_1|)^2 \to c^2$ as $x \to x_I$ for $I = 0, 1$, where c is a constant related to the periodicity of ϕ_1. Explicitly, in terms of the proper distance s, this smoothness condition is $\partial_s \sqrt{P} \to \pm c$ which implies $\sqrt{P}$ has simple zeros at the endpoints (just like $\sqrt{Q}$). It follows that P/Q is a smooth positive function on S. To treat this case it is thus convenient to define a smooth function F which is strictly positive on S by

$$P = \frac{Q}{\Gamma F} \tag{16}$$

in terms of which the horizon geometry is

$$g = \frac{\Gamma}{Q}\mathrm{d}x^2 + \frac{Q}{\Gamma F}(\mathrm{d}\phi^1 + \omega \mathrm{d}\phi^2)^2 + F(\mathrm{d}\phi^2)^2 \tag{17}$$

and equation (8) becomes

$$\Gamma P^2 \omega'^2 = Q'' - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q\Gamma'}{\Gamma} + \frac{QF'}{F}\right) + 2\Lambda\Gamma + \frac{P}{\Gamma} \tag{18}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{QF'}{F}\right) - 2\Lambda\Gamma \tag{19}$$

where in the second equality we used (5) and (7). Now, noting that $\omega = \eta_1 \cdot \eta_2 / |\eta_1|^2$, we deduce that ω is a smooth function on S (as shown above, in terms of the proper distance s, the denominator has a double zero at the endpoints, whereas the numerator has at least a double zero). But (9) can be written as $k = \Gamma P^2 \omega' = PF^{-1}Q\omega'$ and evaluating this at the endpoints thus implies $k = 0$. Hence $\omega' = 0$ and so (19) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{QF'}{F}\right) + 2\Lambda\Gamma = 0\ . \tag{20}$$

Integrating this over the interval we find the boundary term vanishes leaving

$$2\Lambda \int_{x_0}^{x_1} \Gamma \mathrm{d}x = 0\ . \tag{21}$$

But by definition Γ is strictly positive, so we have found a contradiction unless $\Lambda = 0$. Thus there are no $\Lambda \neq 0$ solutions with ring topology in this case either. For $\Lambda = 0$ equation (20) implies QF'/F is a constant and evaluating this at the endpoints shows that F is a constant; upon removing the conical singularities at the endpoints the resulting regular solution is isometric to a boosted extreme Kerr string horizon (which includes the extreme black ring horizon) [11].

Therefore, we have proved that there are no extreme horizons with $S^1 \times S^2$ topology and $\Lambda \neq 0$, in particular excluding the anti-de Sitter case $\Lambda < 0$. It is worth noting that our proof also rules out the possibility of an extreme AdS/dS black ring held in equilibrium by a conical singularity, i.e., the above proof remains valid even if one can not simultaneously remove the conical singularities at the endpoints $x = x_I$.[3] In contrast, for supersymmetric AdS solutions there is a ring horizon with a conical singularity [6].

It would be interesting to complete the classification of locally S^3 extreme horizons in the $\Lambda \neq 0$ case; a natural conjecture is that the known solution (corresponding to linear $\Gamma(x)$) is the most general. Furthermore, given the absence of both vacuum and supersymmetric anti-de Sitter black rings, it is now tempting to conjecture the absence of generic extreme charged anti-de Sitter black rings in Einstein-Maxwell type theories (in particular including minimal gauged supergravity), at least with biaxial symmetry.

Acknowledgments. The author is supported by STFC [ST/L000458/1].

[3]More precisely, if v is the vector which vanishes at the endpoints, then $(\mathrm{d}|v|)^2 \to (c_I)^2$ as $x \to x_I$ where $c_0 - c_1 \neq 0$ is proportional to the angular deficit.

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